Successive Differentiation

1. Contents: - a. Introduction to successive differentiation in terms of finding n^{th} order derivatives of standard functions viz.

e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$, $(ax+b)^m$, $(ax+b)^{-m}$, a^{ax+b}, $e^{ax}\sin(bx+c)$, $e^{ax}\cos(bx+c)$, $\log(ax+b)$ etc.

b. Leibniz's theorem (without proof) and problems based on it.

2. Sample Problems: based on Direct Differentiation of n^{th} order

1) Find the n^{th} order derivative of sinx sin3x

solu: *Let* $y = \sin x \sin 3x$

$$y = \frac{1}{2} 2\sin x \sin 3x$$

$$y = \frac{1}{2}[\cos(x-3x) - \cos(x+3x)]$$

$$y = \frac{1}{2}[\cos(2x) - \cos(4x)]$$

$$y_n = \frac{1}{2}[2^n \cos(2x + n\frac{\pi}{2}) - 4^n \cos(4x + n\frac{\pi}{2})]$$

2) Find the n^{th} order derivative of $e^{5x}\cos x \cos 3x$

solu: *Let* $y = e^{5x}\cos x \cos 3x$

$$y = e^{5x}\frac{1}{2}[\cos(x-3x) + \cos(x+3x)]$$

$$y = e^{5x}\frac{1}{2}[\cos(-2x) + \cos(4x)]$$

$$y = e^{5x}\frac{1}{2}[\cos(2x) + \cos(4x)]$$

$$y = \frac{1}{2}[e^{5x}\cos(2x) + e^{5x}\cos(4x)]$$

$$y_n = \frac{1}{2}[(5^2+2^2)^{\frac{n}{2}} e^{5x}\cos(2x + n\frac{\pi}{2}) + (5^2+4^2)^{\frac{n}{2}} e^{5x}\cos(4x + n\frac{\pi}{2})]$$

$$y_n = \frac{1}{2}[(29)^{\frac{n}{2}} e^{5x}\cos(2x + n\frac{\pi}{2}) + (41)^{\frac{n}{2}} e^{5x}\cos(4x + n\frac{\pi}{2})]$$

3) *If* $y = \tan^{-1}\left(\frac{x}{a}\right)$, *then prove that* $y_n = \frac{(-1)^{n-1}(n-1)!}{a^n}\sin n\theta \sin^n\theta$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$.

solu: *Given that* $y = \tan^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow y_1 = \frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{1}{a}$$

$$y_1 = \frac{a^2}{a^2+x^2}\frac{1}{a} \Rightarrow y_1 = \frac{a}{a^2+x^2} \Rightarrow y_1 = \frac{a}{(x+ai)(x-ai)}$$

$$y_1 = \frac{2ai}{2i(x+ai)(x-ai)} \Rightarrow y_1 = \frac{(x+ai)-(x-ai)}{2i(x+ai)(x-ai)}$$

$$y_1 = \frac{1}{2i}\left[\frac{1}{(x-ai)} - \frac{1}{(x+ai)}\right]$$

differantiate it $(n-1)$ *times with respect* x

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(1+n-1-1)!}{(1-1)!(x-ai)^{1+n-1}}1^{n-1} - \frac{(-1)^{n-1}(1+n-1-1)!}{(1-1)!(x+ai)^{1+n-1}}1^{n-1}\right]$$

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-ai)^n} - \frac{1}{(x+ai)^n}\right]$$

put $x = r\cos\theta$ *and* $a = r\sin\theta$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(r\cos\theta - ir\sin\theta)^n} - \frac{1}{(r\cos\theta + ir\sin\theta)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\sin\theta)^n} - \frac{1}{r^n(\cos\theta + i\sin\theta)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[\frac{1}{(\cos\theta - i\sin\theta)^n} - \frac{1}{(\cos\theta + i\sin\theta)^n}\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[\left(e^{i\theta}\right)^n - \left(e^{-i\theta}\right)^n\right]$$

$$y_n = \frac{(-1)^{n-1}(n-1)!}{r^n}\left[\frac{e^{in\theta} - e^{-in\theta}}{2i}\right] \Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$\sin ce\ a = r\sin\theta \Rightarrow \frac{1}{r} = \frac{\sin\theta}{a} \Rightarrow \frac{1}{r^n} = \frac{\sin^n\theta}{a^n}$$

$$\therefore y_n = (-1)^n(n)!\frac{\sin^n\theta}{a^n}\sin n\theta$$

3. Sample Problems: based on Leibniz's theorem

4) Find the n[th] derivative of $x^2 \sin x$

solu: $Let\ y = x^2 \sin x$

$$y_n = \frac{d^n}{dx^n}(x^2 \sin x)$$

then by leibnitz's theorem

$$y_n = {}^n c_o(\frac{d^n}{dx^n}\sin x)(x^2) + {}^n c_1(\frac{d^{n-1}}{dx^{n-1}}\sin x)(\frac{d}{dx}x^2) + {}^n c_2(\frac{d^{n-2}}{dx^{n-2}}\sin x)(\frac{d^2}{dx^2}x^2)$$

$$y_n = \sin(x + n\frac{\pi}{2})(x^2) + n\sin(x + (n-1)\frac{\pi}{2})(2x) + \frac{n(n-1)}{2}\sin(x + (n-2)\frac{\pi}{2})(2)$$

$$y_n = \sin\{x + n\frac{\pi}{2}\}(x^2) + 2nx\sin\{x + (n-1)\frac{\pi}{2}\} + n(n-1)\sin\{x + (n-2)\frac{\pi}{2}\}$$

5). *If* $y = a\cos(\log x) + b\sin(\log x)$, *show that*

$$x^2 y_{n+2} + (2n+1)x\, y_{n+1} + (n^2+1)\, y_n = 0.$$

Solu: *Given that* $y = a\cos(\log x) + b\sin(\log x)$

differentiate it with respect to x

$$y_1 = -a\sin(\log x)\frac{1}{x} + b\cos(\log x)\frac{1}{x}$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

differentiate it with respect to x

$$xy_2 + y_1 = -a\cos(\log x)\frac{1}{x} - b\sin(\log x)\frac{1}{x}$$

$$x^2 y_2 + xy_1 = -\{a\cos(\log x) + b\sin(\log x)\}$$

$$x^2 y_2 + xy_1 = -y$$

differentiate it with respect to x, n times using $L.T.$

$$\frac{d^n}{dx^n}(x^2 y_2 + xy_1) = -y_n$$

$$\{{}^n c_0 y_{n+2} x^2 + {}^n c_1 y_{n+1}(2x) + {}^n c_2 y_n(2)\} + \{{}^n c_0 y_{n+1} x + {}^n c_1 y_n\} = -y_n$$

$$\{{}^n c_0 y_{n+2} x^2 + {}^n c_1 y_{n+1}(2x) + {}^n c_2 y_n(2)\} + \{{}^n c_0 y_{n+1} x + {}^n c_1 y_n\} = -y_n$$

$$y_{n+2}x^2 + 2nxy_{n+1} + n(n-1)y_n + y_{n+1}x + ny_n = -y_n$$

$$y_{n+2}x^2 + (2n+1)xy_{n+1} + n(n-1)y_n + ny_n + y_n = 0$$

$$y_{n+2}x^2 + (2n+1)xy_{n+1} + \{n(n-1) + n + 1\}y_n = 0$$

$$y_{n+2}x^2 + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

I[st] Exercise: Direct Differentiation of n[th] order

1 . *Find* n^{th} *order derivatives of* :

$(i)\ \cos x\cos 2x\cos 3x$ $\quad(ii)\ \sin^5 x\cos^3 x$

$(iii)\ \cos^4 x$ $\quad(iv)\ e^{x\cos\alpha}\cos(x\sin\alpha)$

$(v)\ e^x(\sin x+\cos x)$ $\quad(vi)\ \sin 2x\sin 3x\cos 4x$

Ans.: $(i)\ 2^{n-2}\cos\left(2x+\frac{n\pi}{2}\right)+4^{n-1}\cos\left(4x+\frac{n\pi}{2}\right)+\frac{6^n}{4}\cos\left(6x+\frac{n\pi}{2}\right)$

$(ii)\ 2^{3n-7}\sin\left(8x+\frac{n\pi}{2}\right)-3^n\,2^{n-6}\sin\left(6x+\frac{n\pi}{2}\right)-2^{2n-6}\sin\left(4x+\frac{n\pi}{2}\right)$
$+3\cdot 2^{n-6}\sin\left(2x+\frac{n\pi}{2}\right)$

$(iii)\ 2^{n-1}\cos\left(2x+\frac{n\pi}{2}\right)+2^{2n-3}\cos\left(4x+\frac{n\pi}{2}\right)$

$(iv)\ e^{x\cos\alpha}\cos(n\alpha+x\sin\alpha)$

$(v)\ 2^{\frac{n+1}{2}}e^x\sin\left(x+(n+1)\frac{\pi}{4}\right)$

$(vi)\ \frac{1}{4}\left[5^n\cos\left(5x+\frac{n\pi}{2}\right)+3^n\cos\left(3x+\frac{n\pi}{2}\right)-9^n\cos\left(9x+\frac{n\pi}{2}\right)-\cos\left(x+\frac{n\pi}{2}\right)\right]$

2. *If* $y=\sin px+\cos px$, *then prove that*

$y_n=p^n\left\{1+(-1)^n\sin 2px\right\}^{\frac{1}{2}}$. *Find* $y_8(\pi)$ *where* $p=\frac{1}{4}$

Ans: $\left(1/2\right)^{\frac{31}{2}}$

3. *If* $y=(x-1)^n$, *prove that* $y+\frac{y_1}{1!}+\frac{y_2}{2!}+\frac{y_3}{3!}+....+\frac{y_n}{n!}=x^n$.

4. *Find* n^{th} *derivatives of* :

$(i)\ y=\frac{x^4}{(x-1)(x-2)}$ $\quad(ii)\ y=\frac{x^2+4x+1}{x^3+2x^2-x-2}$ $\quad(iii)\ y=\frac{1}{1+x+x^2}$

Ans.: (i) $(-1)^n n!\left[\frac{16}{(x-2)^{n+1}}-\frac{1}{(x-1)^{n+1}}\right],\ n>2$

(ii) $(-1)^n n!\left[\frac{1}{(x+1)^{n+1}}+\frac{1}{(x-1)^{n+1}}-\frac{1}{(x+2)^{n+1}}\right]$ (iii) $\frac{(-1)^n\ n!\ 2^{n+2}\sin^{n+1}\theta\ \sin(n+1)\theta}{3^{\frac{n+2}{2}}}$

5. *Find* y_n *where* $y = \dfrac{8x}{x^3 - 2x^2 - 4x + 8}$

6. *Show that* $\dfrac{d^4}{dx^4}\left[\dfrac{x^3}{x^2-1}\right]_{x=0} = 0.$

7. *If* $y = \dfrac{x^3}{x^2-1}$ *then* $(y_n)_0 = \begin{cases} -(n!) \text{ if } n-odd \\ 0 \text{ if } n-even. \end{cases}$

8. *If* $y = \dfrac{x}{x^2+a^2}$, *prove that* $y_n = \dfrac{(-1)^n n!}{a^{n+1}} (\sin\theta)^{n+1}\cos(n+1)\theta.$

9. *If* $y = \dfrac{1}{x^2+a^2}$, *then prove that* $y_n = \dfrac{(-1)^n n! \sin(n+1)\theta \ (\sin\theta)^{n+1}}{a^{n+2}}$

where $\theta = \tan^{-1}\left(\dfrac{a}{x}\right).$

10. *If* $y = \dfrac{1}{1+x+x^2}$ *then prove that* $y_n = \dfrac{2(-1)^n}{\sqrt{3}}\dfrac{n!}{r^{n+1}}\sin(n+1)\theta$

where $\theta = \cot^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$ *and* $r = \sqrt{1+x+x^2}$

11. *Show that* $\dfrac{d^n}{dx^n}(\tan^{-1}x) = (-1)^{n-1}(n-1)!\dfrac{\sin\left(n\tan^{-1}\dfrac{1}{x}\right)}{(x^2+1)^{\frac{n}{2}}}.$

12. *If* $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ *find* n^{th} *differential coefficient, then convert it into polar form.*

Ans: $2(-1)^{n-1}(n-1)!(\sin\theta)^n \sin n\theta, \quad \theta = \tan^{-1}\left(\dfrac{1}{x}\right)$

13. *Find* y_n *where* (*i*) $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ (*ii*) $y = \cos^{-1}\left(\dfrac{x-x^{-1}}{x+x^{-1}}\right)$

Ans: (*i*) $2(-1)^{n-1}(n-1)!(\sin\theta)^n \sin n\theta, \quad \theta = \tan^{-1}\left(\dfrac{1}{x}\right)$

(*ii*) $\dfrac{2(-1)^{n-1}(n-1)!\sin n\theta}{(x^2+1)^{\frac{n}{2}}}, \quad \theta = \tan^{-1}\left(\dfrac{1}{x}\right)$

14. *If* $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2}\;-1}{x}\right]$, *prove that* $y_n = \dfrac{1}{2}(-1)^{n-1}\,(n-1)\,!\sin^n\theta\sin n\theta$,

$\theta = \cot^{-1} x$.

15. *If* $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right)$, *prove that* $y_n = (-1)^{n-1}\,(n-1)!\,\sin^n\theta\,\sin n\theta$,

where $\theta = \tan^{-1}\dfrac{1}{x}$.

16. *If* $y = x\log(1+x)$, *then prove that* $y_n = \dfrac{(-1)^{n-2}\,(n-2)!(x+n)}{(x+1)^n}$.

17. *If* $y = x\,\log\left[\dfrac{(x-1)}{(x+1)}\right]$, *prove that*

$$y_n = (-1)^n\,(n-2)\,!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right] \quad if\ \ n \ge 2.$$

18. *Show that* $y_n = \dfrac{(-1)^n\,(n-2)!}{2}\left[\dfrac{x+n}{(x+1)^n} - \dfrac{x-n}{(x-1)^n}\right]$, $n \ge 2$

if $y = x\,\cot h^{-1} x$.

19. *If* $I_n = \dfrac{d^n}{dx^n}(x^n\log x)$, *prove that* $I_n = n\,I_{n-1} + (n-1)\,!$.

Hence show that $I_n = n!\left(\log x + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dots + \dfrac{1}{n}\right)$.

20. Prove that $\dfrac{d^n}{dx^n}\left\{x^{n-1}e^{\frac{1}{x}}\right\} = \dfrac{(-1)^n\,e^{\frac{1}{x}}}{x^{n+1}}$

<u>IInd Exercise</u> : Leibnitz's Theorem

1. *Find* n^{th} *derivative of*

(*i*) $x^3\cos x$

(*ii*) $e^x\,(2x+3)^3$

(*iii*) $x^2\,e^{ax}$

(*iv*) $x^2\tan^{-1}x$

(*v*) $x^2\,e^x\,\cos x$

(*vi*) $x^2\,e^x$

(*vii*) $2^x\sin^2 x\cos^3 x$

(*viii*) $e^{2x}\cos^2 x\sin x$

(*ix*) $2^x\cos^9 x$.

Ans: (*i*) $x^3\cos\left(x+n\dfrac{\pi}{2}\right) + 3nx^2\cos\left(x+\dfrac{1}{2}(n-1)\pi\right) + 3x\,n(n-1)\cos\left(x+\dfrac{1}{2}(n-2)\pi\right)$

$+\,n(n-1)(n-2)\cos\left(x+\dfrac{1}{2}(n-3)\pi\right)$.

$(ii)\ e^{x}\left\{(2x+3)^{3}+6n(2x+3)^{2}+12n(n-1)(2x+3)+8n(n-1)(n-2)\right\}$

$(iii)\ e^{ax}\left\{a^{n}x^{2}+2nxa^{n-1}+n(n-1)a^{n-2}\right\}$

$$(iv)\ \frac{x^{2}(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(x-i)^{n}}-\frac{1}{(x+i)^{n}}\right]+\frac{2nx(-1)^{n-2}(n-2)!}{2i}\left[\frac{1}{(x-i)^{n-1}}-\frac{1}{(x+i)^{n-1}}\right]$$
$$+\frac{n(n-1)(-1)^{n-3}(n-3)!}{2i}\left[\frac{1}{(x-i)^{n-2}}-\frac{1}{(x+i)^{n-2}}\right]$$

$$(v)\ e^{x}\left(2^{\frac{n}{2}}\cos\left(x+\frac{n\pi}{4}\right)+2nx2^{\frac{n-1}{2}}\cos\left(x+\frac{(n-1)\pi}{4}\right)+n(n-1)2^{\frac{n-2}{2}}\cos\left(x+\frac{(n-2)\pi}{4}\right)\right)$$

$(vi)\ e^{x}\left\{x^{2}+2nx+n(n-1)\right\}$

$$(vii)\ (\log 2)^{n}e^{x\log 2}\frac{1}{16}\left(2\cos\left(\frac{\pi}{2}+x\right)-5\cos\left(\frac{\pi}{2}+5x\right)-3\cos\left(\frac{\pi}{2}+3x\right)\right)$$
$$+n(\log 2)^{n-1}e^{x\log 2}\frac{1}{16}\left(2\cos\left(\frac{2\pi}{2}+x\right)-5^{2}\cos\left(\frac{2\pi}{2}+5x\right)-3^{2}\cos\left(\frac{2\pi}{2}+3x\right)\right)+---$$
$$---+e^{x\log 2}\frac{1}{16}\left(2\cos\left(\frac{n\pi}{2}+x\right)-5^{n}\cos\left(\frac{n\pi}{2}+5x\right)-3^{n}\cos\left(\frac{n\pi}{2}+3x\right)\right)$$

$$(viii)\ \frac{e^{2x}}{4}2^{n}\left(\sin 3x+\sin x\right)+\frac{ne^{2x}}{4}2^{n-1}\left(3\sin\left(3x+\frac{\pi}{2}\right)+\sin\left(x+\frac{\pi}{2}\right)\right)+----$$
$$----+\frac{e^{2x}}{4}\left(3^{n}\sin\left(3x+\frac{n\pi}{2}\right)+\sin\left(x+\frac{n\pi}{2}\right)\right)$$

2. *If* $y=\tan^{-1}\left(\frac{a+x}{a-x}\right)$, *then prove that*

$$(a^{2}+x^{2})\ y_{n+2}+2(n+1)x\,y_{n+1}+n(n+1)\ y_{n}=0.$$

3. *If* $x=\cosh\left(\frac{1}{m}\log y\right)$, *then*

$$(x^{2}-1)\,y_{n+2}+(2n+1)x\ y_{n+1}+(n^{2}-m^{2})\,y_{n}=0.$$

4. *If* $y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$ *show that* $(x^{2}-1)y_{n+2}+(2n+1)xy_{n+1}+(n^{2}-m^{2})y_{n}=0$

5. *If* $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^{n}$, *prove that*

$$x^{2}\ y_{n+2}+(2n+1)x\ y_{n+1}+2n^{2}\ y_{n}=0.$$

6. *If* $y=\sec^{-1}x$, *prove that*

$$x(x^{2}-1)\,y_{n+2}+\left[(2+3n)\,x^{2}-(n+1)\,y_{n+1}\right]+n(3n+1)\,x\ y_{n}+n^{2}(n-1)\,y_{n-1}=0.$$

7. *If* $y=\left(x^2-1\right)^n$, *prove that* $(x^2-1)\,y_{n+2}+2x\,y_{n+1}-n(n+1)\,y_n=0.$

8. *If* $y=x^2e^x$ *then show that*

$$y_n=\frac{1}{2}n(n-1)y_2-n(n-2)y_1+\frac{1}{2}(n-1)(n-2)y$$

9. *If* $y=\sin\left[\log\,(x^2+2x+1)\right]$, *prove that*

$$(x+1)^2\,y_{n+2}+(2n+1)\,(x+1)\,y_{n+1}+(n^2+4)\,y_n=0.$$

10. *If* $y=\cot x$, *prove that*

$$nc_1\,y_{n-1}(0)-nc_3\,y_{n-3}(0)+nc_5\,y_{n-5}(0)-.....=\cos\frac{n\pi}{2}.$$

11. *If* $f(x)=\tan x$, *then prove that*

$$f^n(0)-nc_2\,f^{n-2}(0)+nc_4\,f^{n-4}(0)-.....=\sin\frac{n\pi}{2}.$$

12. *If* $y=e^{m\sin^{-1}x}$, *prove that* $(1-x^2)\,y_{n+2}-(2n+1)\,x\,y_{n+1}-(n^2+m^2)\,y_n=0.$

13. *If* $y=e^{m\cos^{-1}x}$, *show that* $(1-x^2)\,y_{n+2}-(2n+1)\,x\,y_{n+1}-(n^2+m^2)\,y_n=0$ *and hence find* $y_n(0)$.

Ans: $$(y_n)_0=\begin{cases} m^2\,(2^2+m^2)\,(4^2+m^2)....((n-2)^2+m^2)\,e^{\frac{m\pi}{2}}, & \text{for 'n' even}\\ -m\,(1^2+m^2)\,(3^2+m^2)....((n-2)^2+m^2)\,e^{\frac{m\pi}{2}}, & \text{for 'n' odd}\end{cases}$$

14. *If* $y=\tan^{-1}x$, *then show that* $(x^2+1)\,y_{n+1}+2n\,x\,y_n+n(n-1)\,y_{n-1}=0$

and $$y_n(0)=\begin{cases}0 & \text{if } n-\text{even.}\\ (-1)^{\frac{n-1}{2}}(n-1)! & \text{if } n-\text{odd.}\end{cases}$$

15. *If* $y=(\sin^{-1}x)^2=a_0+a_1x+a_2x^2+.......$, *prove that*

$$(n+1)(n+2)\,a_{n+2}=n^2\,a_n.$$

16. *If* $y=\left(x+\sqrt{1+x^2}\right)^m$, *prove that*

(*i*) $(y_{2n})_0=\left[m^2-(2n-2)^2\right]\left[m^2-(2n-4)^2\right]......\left[m^2-2^2\right]m^2.$

(*ii*) $(y_{2n+1})_0=\left[m^2-(2n-1)^2\right]\left[m^2-(2n-3)^2\right]......\left[m^2-1^2\right]m.$

17. *If* $y=\sinh^{-1}x$, *then prove that* $(1+x^2)\,y_{n+2}+(2n+1)\,x\,y_{n+1}+n^2\,y_n=0.$

18. *If* $y=\log\left(x+\sqrt{x^2+1}\right)$, *prove that* $y_{2n}(0)=0$

& $y_{2n+1}(0)=(-1)^n\left[1^2.3^2.5^2.......(2n-1)^2\right].$

19. *If* $y = (\sin^{-1} x)^2$ *prove that* $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$ *and*

$y_{2n+1}(0) = 0$ *and* $y_{2n}(0) = 2^{2n-1}\left[(n-1)!\right]^2$.

20. *If* $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ *, prove that* $(1-x^2)\,y_{n+1} - (2n+1)\,x\,y_n - n^2\,y_{n-1} = 0.$

21. *If* $y = \sin\left(m \sin^{-1} x\right)$*, then prove that*

(*i*) $(1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} - (n^2-m^2)\,y_n = 0.$

(*ii*) $(y_n)_0 = \begin{cases} \left\{(n-2)^2-m^2\right\}\left\{(n-4)^2-m^2\right\}.....(1-m^2)\,m\ , & for\ 'n'\ odd \\ 0\ , & for\ 'n'\ even. \end{cases}$

22. *If* $y = \left[\log\left(x + \sqrt{1+x^2}\right)\right]^2$*, show that* $y_{n+2}(0) = -n^2\,y_n(0).$

23. *If* $y = x^2 e^{2x}$*, prove that* $y_n(0) = 2^{n-2}\,n(n-1).$

24. *If* $y = \sqrt{\dfrac{1+x}{1-x}}$ *, prove that* $y = (1-x^2)\,y_1$ *and hence prove that*

$(1-x^2)\,y_n - \left[2(n-1)x + 1\right]y_{n-1} - (n-1)(n-2)\,y_{n-2} = 0.$

25. *If* $y = x^n \log x$*, then prove that* $y_{n+1} = \dfrac{n!}{x}.$

Learning Resources :

- Higher Engineering Mathematics, Dr. B. S. Grewal, Khanna Publications.
- Advanced Engineering Mathematics, Erwin Kreyszing, Wiley Eastern Limited, 8th Ed.